# Analysis and Experiments on Stress Waves in Planar Trusses

S. M. Howard and Y-H Pao

# ANALYSIS AND EXPERIMENTS ON STRESS WAVES IN PLANAR TRUSSES

By Samuel M. Howard[1] and Yih-Hsing Pao[2]

**ABSTRACT:** The dynamics of planar trusses are investigated in terms of axial (longitudinal) stress waves, which propagate along structural members and scatter at the joints. The scattering coefficients representing the reflection and transmission of axial waves at each joint are derived from the dynamics and compatibility conditions of the joint. The complex multiple reflections of waves within the structure are evaluated in the frequency domain with a newly developed reverberation matrix, which is formulated from the scattering coefficients and propagating phase factors. Transient waves are then derived by Fourier synthesis, and evaluated by a Fast Fourier Transform algorithm. Experimental results of propagating broad band pulses are presented for a truss model excited by a step loading. Comparison between theoretical results and transient wave records indicate that the axial wave theory is valid only for the response at the very early time. The discrepancy is much reduced if the scattering coefficients are modified to allow mode conversion from axial to flexural waves at the joint.

## INTRODUCTION

The study of impact loading on truss structures dates as far back as the turn of this century when large railroad bridges were being built to carry heavy locomotives. Some of the early research work was mentioned by Nowacki (1963). Boley and Chao (1957) studied how stress waves are generated by impact force at a joint of a truss, propagate along the members emanating from the loaded joint, and scatter from the joints at opposite ends. Furthermore, they noted that by calculating all possible paths of propagation from joint to joint, the dynamic response of a truss can be described as a superposition of waves that reverberate within a truss. Because of the numerous different paths that waves take throughout a typical truss, the calculation of all of these reverberations is indeed very complicated. Lacking an electronic computer at that time, they calculated only the first two or three reverberations in the example.

Recently, there has been a revival of interest in the wave propagations or vibrations of lattice-type structures for the purpose of controlling localized disturbances or monitoring structural integrity (Flotow 1986a,b; Nagem and Williams 1989). This paper reports an analytical method for determining transient responses of a truss, and the experimental observations of stress waves propagating in a laboratory model. In the theory, it is assumed, as in Boley and Chao (1957), that the members of a truss support only axial forces, and hence uniform axial stresses at each cross section. This critical assumption arises from the static theory of pin-jointed trusses, which are known to support only axial forces in their constituent members. Additionally, the assumption of only axial forces is commonly used in static analysis even when the members are riveted to gusset plates of the structure or connected by welding, provided that the members are sufficiently long and slender but would not buckle under the applied load (Parcel and Moorman 1955). In the experimental model, all structural members were, however, welded together, because the writers encountered great difficulty in fabricating pin-connectors that would keep all members aligned to the same plane. Since the inelastic effect of friction and slack at the joint need not be considered here, the dynamic responses are assumed linear elastic for the entire structure (Howard 1990).

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As will be shown later in this paper, however, transverse forces play a significant role in the transient response even when they can be neglected in the statical analysis. They arise due to mode conversion when an incident axial wave is scattered at a joint into bending (transverse) waves, which are reconverted into axial waves at the next joint. Interestingly, our experiments indicate that much of the early transient behavior of a welded truss can be described in terms of the axial waves alone, provided that the effect of bending is included in the evaluation of the scattering coefficients of these joints. It is hoped that this limited, but simpler, theory involving only axial motion gives information about the early transients that may be useful in applications of active control, system identification, or nondestructive evaluation of structures.

## DEFINITIONS

Consider a planar truss with $n$ joints and $m$ members. Each joint is identified with a number 1 through $n$ and each member by the two numbers of its end joints. In the analysis, joints will be denoted with capital letters alphabetically between $I$, $J$, ... $Q$ and members with the two capital letters representing the joints at both ends. Physical quantities associated with joint $J$ or with member $JK$ will carry superscripts. For example, $\mathbf{f}^J$ denotes a vector force applied at joint $J$, and $u^{JK}$ denotes the axial displacement in member $JK$. The symbol $n^J$ will denote the number of neighboring joints connected by $m^J$ members to joint $J$ ($m^J = n^J$).

Fig. 1 depicts joint geometry. All joints at the undeformed positions are referred to a fixed Cartesian frame with coordinates $X$, $Y$. In addition a set of local coordinates $x$, $y$ is introduced for each member connected to joint $J$. Thus $x^{JK}$ is the coordinate that originates from joint $J$ along the center line of member $JK$, and $x^{KJ}$ is another coordinate along the same line but originating at joint $K$.

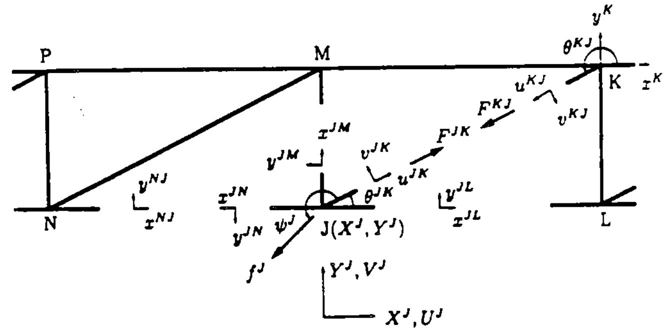


FIG. 1. Joint Geometry and Notation

## AXIAL WAVES IN STRUCTURAL MEMBERS

For members $JK$ of a truss, the axial displacement $u^{JK}(x^{JK}, t)$ obeys the following well-known wave equation:

$$\frac{\partial^2 u^{JK}(x^{JK}, t)}{\partial x^{JK2}} = \frac{1}{c^2}\frac{\partial^2 u^{JK}(x^{JK}, t)}{\partial t^2}; \quad c^2 = E/\rho \tag{1}$$

where $E$ = Young's modulus; and $\rho$ = density. The general solution to (1) is

$$u^{JK}(x^{JK}, t) = A^{JK}(t + x^{JK}/c) + D^{JK}(t - x^{JK}/c) \tag{2}$$

where $A$ and $D$ = arbitrary twice differentiable functions.

Eq. (2), which is the D'Alembert solution of (1), may be interpreted as saying that the stresses in a member may be expressed as the superposition of two traveling waves, one of which is traveling in a direction such that it can be said to be arriving at joint $J$("$A$"), and the other of which may be interpreted as departing from joint $J$("$D$"). In an actual truss subjected to loads, $A$ and $D$ are determined by the forces applied at the joints at either end of a member. Upon taking the Fourier transform of (2) the solution for harmonic waves is obtained:

$$u^{JK}(x^{JK}, \omega) = a^{JK}(\omega)\exp[i(kx)^{JK}] + d^{JK}(\omega)\exp[-i(kx)^{JK}] \tag{3}$$

where $\omega$ = radial frequency; $a^{JK}$, $d^{JK}$ = Fourier transforms of $A^{JK}$ and $D^{JK}$; and the wave number $k = \omega/c$.

## DYNAMIC LOADING AND SCATTERING OF WAVES AT JOINT

In order to introduce the subject of wave scattering at a joint, axial waves only are considered, and the contribution of bending forces is addressed later. The joint is treated as an idealized frictionless pin joint without any slack. It is assumed that two types of equations describe the dynamics of the joint: (1) balance of forces; and (2) compatibility of motion of the members (i.e., the motion of the ends of all members must be consistent with the fact they must remain joined together at a joint). As shown by Boley and Chao and discussed further in Appendix I, these equations may be solved and cast into the following succinct form:

$$d^{JM}(\omega) = \sum_Q S^{MJQ} a^{JQ}(\omega) + s^{JM}(\omega) \; (Q = 1^J, 2^J, \ldots, m^J) \tag{4}$$

where the sum over $Q$ = sum over all of the members attached to the joint $J$; and $a^{JQ}(\omega)$ = complex frequency-dependent amplitude of the axial wave arriving at joint $J$ along member $JQ$. Similarly $d^{JM}(\omega)$ represents the wave in member $JM$ that is departing or traveling away from the joint. $S^{MJQ}$, the scattering factor for the joint, is obtained by solving the aforementioned force and compatibility equations for the particular joint geometry. The source term, $s^{JM}(\omega)$, records that portion of departing waves that arise due to external forces applied to the joint. Formulas for $S^{MJQ}$ and $s^{JM}(\omega)$ are given in Appendix I.

## REVERBERATION OF WAVES IN TRUSS

The scattering formulas for each joint may be used to compute the transient behavior of an entire truss or frame. This may be done in either the time or frequency domains. The time domain approach requires a "ray tracing" procedure for all possible reverberation paths throughout the truss, which can prove quite cumbersome (Howard 1990). The computational complexity of this approach quickly overwhelms even large digital computers. For if $m^J$ is the number of members attached at a typical joint, the number of ray paths to be considered to model $N$ reverberations is $(m^J)^N$. Solution by ray tracing becomes even more cumbersome if one ever wants to consider scattering coefficients that exhibit frequency dispersion (as in a later section of this paper) because then each ray's path will involve a different sequence of scattering coefficients, each requiring a separate computation of an inverse Fourier Transform to calculate the transient. To overcome the excessive computational demands of ray tracing, a more efficient technique was sought for calculating waves in a truss. A new method, here called the reverberation method, was developed for this purpose.

First, combine (4) into a matrix expressing the relations between all arriving and departing waves. For example, for the joint depicted in Fig. 1 this matrix is as follows:

$$\begin{bmatrix} \mathbf{d}^{JK} \\ \mathbf{d}^{JL} \\ \mathbf{d}^{JM} \\ \mathbf{d}^{JN} \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{KJK} & \mathbf{S}^{KJL} & \mathbf{S}^{KJM} & \mathbf{S}^{KJN} \\ \mathbf{S}^{LJK} & \mathbf{S}^{LJL} & \mathbf{S}^{LJM} & \mathbf{S}^{LJN} \\ \mathbf{S}^{MJK} & \mathbf{S}^{MJL} & \mathbf{S}^{MJM} & \mathbf{S}^{MJN} \\ \mathbf{S}^{NJK} & \mathbf{S}^{NJL} & \mathbf{S}^{NJM} & \mathbf{S}^{NJN} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{JK} \\ \mathbf{a}^{JL} \\ \mathbf{a}^{JM} \\ \mathbf{a}^{JN} \end{bmatrix} + \begin{bmatrix} \mathbf{s}^{JK} \\ \mathbf{s}^{JL} \\ \mathbf{s}^{JM} \\ \mathbf{s}^{JN} \end{bmatrix} \tag{5}$$

or, more compactly:

$$\mathbf{d}^J = \mathbf{S}^J\mathbf{a}^J + \mathbf{s}^J \tag{6}$$

The column vector $\mathbf{d}^J$ represents a complete list of departures from joint $J$ in all of the members, and $\mathbf{a}^J$ represents that of the arrivals. Here $\mathbf{d}^J$ will be called the total departure vector at joint $J$, and $\mathbf{a}^J$ the total arrival vector at joint $J$. The matrix $\mathbf{S}^J$ is called the scattering matrix at joint $J$, and $\mathbf{s}^J$ the source vector at joint $J$.

For a truss with $n$ joints and $m$ members, $\mathbf{d}^J$ ($J$ = 1, 2, . . . , $n$) in (6) may be treated as a submatrix and a consolidated matrix may be constructed for the entire structure in the following form:

$$\begin{bmatrix} \mathbf{d}^1 \\ \mathbf{d}^2 \\ \vdots \\ \mathbf{d}^n \end{bmatrix} = \begin{bmatrix} \mathbf{S}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{S}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mathbf{S}^n \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^n \end{bmatrix} + \begin{bmatrix} \mathbf{s}^1 \\ \mathbf{s}^2 \\ \vdots \\ \mathbf{s}^n \end{bmatrix} \tag{7}$$

or, more compactly:

$$\mathbf{d} = \mathbf{S}\mathbf{a} + \mathbf{s} \tag{8}$$

The column vector $\mathbf{d}$ represents a complete list of departures from all of the joints in all of the members, and $\mathbf{a}$ represents that of the arrivals for the entire structure. Here $\mathbf{d}$ will be called the global departure vector, and $\mathbf{a}$ the global arrival vector. The matrix $\mathbf{S}$ is called the global scattering matrix, and $\mathbf{s}$ the global source vector.

The size of $\mathbf{d}$, $\mathbf{a}$, and $\mathbf{S}$ in (8) can be determined from the number of members in the structure. If $m^1, m^2, \ldots, m^n$ represent the number of members at joints 1, 2, . . . , $n$, respectively, the number of elements in the column matrices $\mathbf{d}$ and $\mathbf{a}$ should be the sum of $m^J$, which equals $2m$, where $m$ is the total number of members in the structure. The sum is $2m$ because each member is connected to a joint at each of its ends. The matrix $\mathbf{S}$ is therefore of size $2m \times 2m$, with square submatrices arranged along its diagonal; $\mathbf{s}$ is a column vector of size $2m$.

The global vectors $\mathbf{d}$ and $\mathbf{a}$ are further related by a phase shift factor. Whenever a departing wave is generated at one end of a member, it propagates down that member and becomes an arriving wave at the opposite end. Hence, there is a one-to-one correspondence between every departing wave and every arriving wave in a member. Specifically, they are related by a phase shift factor

$$a^{JM} = -\exp^{[-i(kl)^{JM}]} d^{MJ} \tag{9}$$

$$d^{JM} = -\exp^{[+i(kl)^{JM}]} a^{MJ} \tag{10}$$

where $l^{JM}$ = length of member $JM$. Note that the superscript order is reversed for the departures in (9) and (10). This is because the departures are defined at joint $M$, whereas the arrivals are defined at joint $J$ at the opposite end of the member. The change of sign is due to the reversal of coordinate $x$. A new total vector is defined at joint $J$, $\tilde{\mathbf{d}}^J$, and a new global vector, $\tilde{\mathbf{d}}$, lists the departures for the entire structure:

$$\tilde{\mathbf{d}}^J = \begin{bmatrix} \tilde{\mathbf{d}}^{KJ} \\ \tilde{\mathbf{d}}^{LJ} \\ \vdots \\ \tilde{\mathbf{d}}^{NJ} \end{bmatrix} \quad (11)$$

$$\tilde{\mathbf{d}} = \begin{bmatrix} \tilde{\mathbf{d}}^1 \\ \tilde{\mathbf{d}}^2 \\ \vdots \\ \tilde{\mathbf{d}}^n \end{bmatrix} \quad (12)$$

The individual elements of the vector $\tilde{\mathbf{d}}$ are exactly identical to those in $\mathbf{d}$, but the order of indicative indices has been permuted. The components of $\mathbf{d}$ are grouped according to their destination joint rather than their departure joint. We may express this equivalence through a permutation matrix $\mathbf{U}$:

$$\tilde{\mathbf{d}} = \mathbf{U}\mathbf{d} \quad (13)$$

where $\mathbf{U} = 2m \times 2m$ matrix that interchanges the elements of $d^{JK}$ and $d^{KJ}$ in the column vector $\mathbf{d}$. Like all permutation matrices, it consists of many 0 elements with a single element of value 1 in each row and each column. The precise form of $\mathbf{U}$ depends on the scheme by which neighboring joints are numbered. Once a joint numbering scheme is chosen, the placement of the elements $d^{JK}$ within the column vector $\mathbf{d}$ is determined and the permutation matrix is easily found by inspection. If, for example, $d^{JK}$ and $d^{KJ}$ occupy, respectively, the $i$th and $p$th elements in the column vector, then $U_{ip} = U_{pi} = 1$.

In terms of the newly introduced vector, $\tilde{\mathbf{d}}$, the arrival vector in (8) may be written in matrix form

$$\mathbf{a} = \mathbf{P}(\mathbf{L}, \omega)\tilde{\mathbf{d}} \quad (14)$$

where

$$\mathbf{P}(\mathbf{L}, \omega) = \begin{bmatrix} \cdots & 0 & \cdots & 0 \\ 0 & -\exp(-ik^{JM}l^{JM}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots \end{bmatrix} \quad (15)$$

and where $\mathbf{L} = 2m$ column vector containing the length of all members. $\mathbf{P}$ is called the propagation matrix. It is a diagonal matrix of size $2m \times 2m$ containing the phase shift factors of (9) and (10) along its diagonal. Using (8), (13), and (14), a direct relationship is finally obtained between the waves in the structure and the applied force

$$\mathbf{d}(\omega) = [\mathbf{I} - \mathbf{SPU}]^{-1}\mathbf{s}(\omega) \quad (16)$$

The matrix product $\mathbf{SPU}$ is defined as the reverberation matrix $\mathbf{R}(\omega)$ (size $2m \times 2m$) and (16) is rewritten as

$$\mathbf{d}(\omega) = [\mathbf{I} - \mathbf{R}(\omega)]^{-1}\mathbf{s}(\omega) \quad (17)$$

The factor $\mathbf{s}(\omega)$ is a source function representing the Fourier transform of the waves in all members generated by the forces $\mathbf{f}^J(t)$ applied at each joint. It may contain many zero elements when the external forces are applied at only one or two joints. The factor $[\mathbf{I} - \mathbf{R}(\omega)]^{-1}$ is the transfer function for the structure, which relates the response of the truss $\mathbf{d}(\omega)$ to the excitation $\mathbf{s}(\omega)$ in the frequency domain. Substitution of an element of $\mathbf{d}(\omega)$ in (17) together with the corresponding element of $\mathbf{a}(\omega) = \mathbf{PUd}(\omega)$ in (14) gives rise to the frequency response (in displacement) for a structure member of the truss.

Since the frequency response of all structure members involves the inverse of the matrix $[\mathbf{I} - \mathbf{R}(\omega)]$, the response is singular if

$$\det[\mathbf{I} - \mathbf{R}(\omega)] = 0 \quad (18)$$

This is the characteristic equation for natural frequencies of the truss.

It appears that the final results shown in (17) etc. resemble those in traditional matrix analysis of structures (Nowacki 1963; Clough and Penzien 1975), where the $[\mathbf{I} - \mathbf{R}]$ resembles the dynamic stiffness matrix and the $[\mathbf{I} - \mathbf{R}]^{-1}$ the dynamic compliance matrix of the entire structure. The physical meanings of the matrices are, however, very much different. This analysis is similar to that presented by Nagam and Williams (1989), who combine the transfer matrix of state variable for the member, the displacement $\mathbf{u}$ and normal force $\mathbf{F}$ for axial waves, with a joint matrix that relates the state variables at one end of the member to those at the joint. Since they were interested in analyzing the vibration of structures, the transfer matrices are formulated in terms of standing waves in the member. The present formulation chooses the departing wave vector $\mathbf{d}$ and the arrival wave vector $\mathbf{a}$ at each end of a member as the state variables. The $\mathbf{d}$ is related to $\mathbf{a}$ by the scattering matrix, and to the applied forces by a joint matrix. An additional equation that relates in two unknown $\mathbf{a}$ and $\mathbf{d}$ is supplemented by considering reverberation of traveling waves within the structure. As shown in the next section, this formulation is particularly suitable for evaluating transient waves by Fourier synthesis.

## TRANSIENT WAVES IN TRUSS

To compute the transient solution for the axial displacement or strain in member $JK$, it is necessary to formulate these quantities in terms of the arriving and departing waves. From (3) and the definition of strain as the derivative of $u^{JK}(x^{JK}, \omega)$ with respect to $x^{JK}$,

$$\varepsilon^{JK}(x^{JK}, \omega) = ik^{JK}(a^{JK}(\omega)\exp\{-i[k(l - x)]^{JM}\} - d^{JK}(\omega)\exp[-i(kx)^{JM}]) \quad (19)$$

To compute the transient solution it is necessary to combine (3) and (19) with (16) and take the inverse Fourier Transform. Then (3) becomes

$$\mathbf{u}(\mathbf{x}, t) = \int_{-\infty}^{\infty} [\mathbf{P}(\mathbf{L} - \mathbf{x}, \omega)\mathbf{U} - \mathbf{P}(\mathbf{x}, \omega)][\mathbf{I} - \mathbf{R}(\omega)]^{-1}\mathbf{s}(\omega)e^{i\omega t}\, d\omega \quad (20)$$

and (19) becomes

$$\boldsymbol{\varepsilon}(\mathbf{x}, t) = \int_{-\infty}^{\infty} i\mathbf{K}[\mathbf{P}(\mathbf{L} - \mathbf{x}, \omega) + \mathbf{P}(\mathbf{x}, \omega)][\mathbf{I} - \mathbf{R}(\omega)]^{-1}\mathbf{s}(\omega)e^{i\omega t}\, d\omega \quad (21)$$

where $\mathbf{x}$ = vector of positions along each of the members; $\mathbf{K}$ = diagonal matrix containing the wave numbers for each member; and $\boldsymbol{\varepsilon}$ = column vector of longitudinal strains corresponding to $\mathbf{u}$.

The evaluation of (20) and (21) is not simple, however, because of the poles associated with the term $[\mathbf{I} - \mathbf{R}(\omega)]^{-1}$. In principle, one may calculate the sum of the residues at each pole of the integrand in (21) to evaluate the integral. This is equivalent to finding the solution in terms of the normal modes of the structure. An accurate transient solution can only be found however, by summing a large number of modes. Another method is to expand the transfer function in a Neumann series

$$[\mathbf{I} - \mathbf{R}(\omega)]^{-1} = (\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \mathbf{R}^3 + \cdots + \mathbf{R}^N) + \mathbf{Q}_N \quad (22)$$

where the remainder $\mathbf{Q}_N$ is given by

$$\mathbf{Q}_N = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{R}^N \quad (23)$$

Substituting (22) into (21) and dropping the remainder, one can then evaluate the inverse transform term by term to obtain the "ray solution"

$$\varepsilon(\mathbf{x}, t) = \int_{-\infty}^{\infty} i\mathbf{K}[\mathbf{P}(\mathbf{L} - \mathbf{x})\mathbf{U} + \mathbf{P}(\mathbf{x})] \times [\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \mathbf{R}^3 + \cdots + \mathbf{R}^N + \mathbf{Q}_N]\mathbf{s}(\omega)e^{i\omega t}\, d\omega \quad (24)$$

The first term in the integral of (24) contains the waves originally generated by the applied forces, which propagate away from the loading joints to the receivers at $\mathbf{x}$. The second term, which is the first term multiplied by $\mathbf{R}(\omega)$, contains the first set of reflections and transmissions of the source waves at the neighboring joints. This process continues up to the $N$th term.

It is important to note that after integration, the transient wave departures represented in each term of the expansion do not occur at the same time. These temporal differences occur because of the different lengths of the truss members and possibly different wave speeds in the members. The phase shift terms in $\mathbf{P}(\mathbf{x}, \omega)$ and $\mathbf{R}(\omega)$ keep track of the various propagation lengths. The maximum initiation time for the reverberations in the truncated series is $t^l_{\max} = Nt_{\max}$, where $t_{\max}$ is the maximum propagation delay of all of the members (i.e., the delay of the longest member when all members are made of the same material). For times greater than $t^l_{\max}$ the response contains no new wave departures.

The truncated series solution is valid up until a time $t^l_{\min}$, beyond which some wave reverberations are not included; $t^l_{\min}$ is equal to $Nt_{\min}$, where $t_{\min}$ is defined as the minimum propagation delay of all of the members. For times greater than $t^l_{\min}$, the neglected terms in the remainder series may have to be included to obtain the correct solution.

## EXPERIMENTS WITH STRUCTURAL MODELS

In designing the experiment, an attempt was made to provide a truss specimen whose behavior was as close as possible to that of a pin-jointed truss. Agreement was first sought between experimental measurements of static strains and those predicted by the pin-jointed model under static loading.

Because it was expected that the joint behavior would be a critical factor in the design of the experiment, considerable effort was expended in selecting an appropriate method of attachment between members. Four different types of joints were created and evaluated, including (a) pinning together aluminum tubes through concentric holes at their ends, in which case the tubes are all held together by a single pin; (b) clamping the members in a style similar to a gusset plate joint, where a pin is inserted through the two clamping plates to hold each member in place; (c) solid joints where the members are continuously joined by machining the truss out of an aluminum plate; and (d) welded joints, which were created by welding together members and filing away excess metal at the joints to minimize the effects of such excess material on wave scattering.

For each specimen, a static load was applied by tying a weight to a joint. Strains were measured using commercially available Constantin foil strain gages which were mounted to each member. Errors for the strain gage readings were estimated to be less than 1 μ strain and were probably dominated by thermal drift. Two nearly identical gages were mounted at opposite sites of each member at midpoint, the average of two readings being the strain due to axial force and one-half of the difference of two readings being the strain due to bending.

Surprisingly, agreement with the static theory was best for the welded aluminum truss shown in Fig. 2. As shown in that figure, the specimen was constructed by welding together 0.635 × 0.635 cm (1/4 in. square rod) bars of *T*-6061 aluminum at the joints. For convenience in constructing the specimen, the dimensions of each bay were chosen to be in a 3:4:5 ratio—specifically, each bay had a height of 0.406 *m* (16 in.), a length of 0.305 m (12 in.), and a diagonal member of length 0.635 m (25 in.). The Young's modulus for the aluminum is taken as 69.0 GPa, and the corresponding bar velocity $c$ = 5,039 m/s. The supports for this specimen were designed to approximate an ideal hinge and an ideal frictionless roller. To accomplish this, at each upper corner of the truss, a 0.28-cm hole was drilled so that support dowels could be inserted. These support dowels were in turn inserted in aluminum brackets, which carried the weight of the structure. One dowel was inserted in a slotted hole in the support brackets, to simulate a roller support.

Two sets of data for axial strains at the middle of each member were recorded, one for a weight of 113 N (25.4 lb) hanging under joint 6, and another for a weight of 140 N (31.5 lb). The measured values agree closely with the theoretical values of strain based on the theory for pin-connected truss, with errors ranging from 0.0 to 11.1%. Theoretical values of strains for the truss with rigid joints, a statically indeterminate structure, were evaluated later (Pao and Keh 1996). They differ not more than 5.7% from those for pin-connected truss. For the lower chord 4-6, vertical member 5-6, and upper chord 5-7 the measured strains ε in micrometer per meter ($10^{-6}$) and the theoretical values (given in parentheses) are, respectively, $\varepsilon_{46}$ = 32.0 (30.6), $\varepsilon_{56}$ = 22.5 (20.4), and $\varepsilon_{57}$ = −30.0 (−30.6) when the weight of 113 N was suspended at joint 6. The cor-

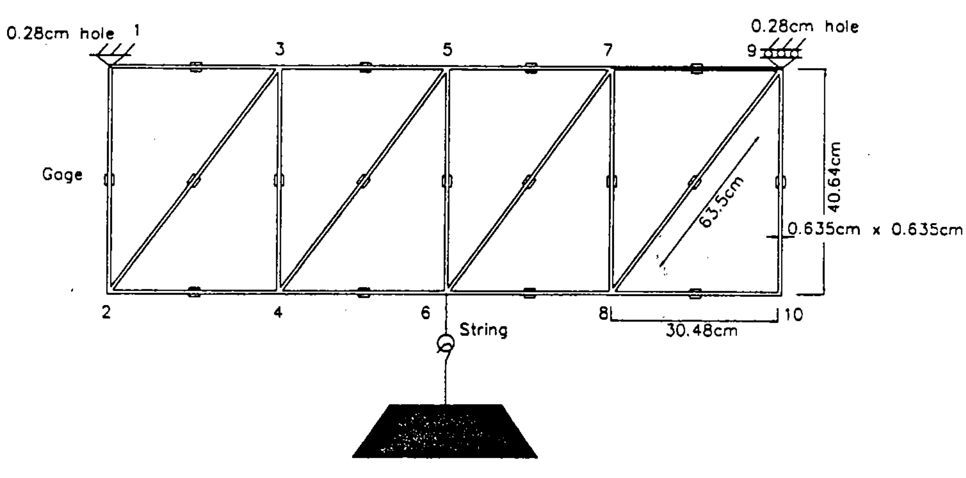


**FIG. 2. Aluminum Truss with Welded Connections**

responding experimental and theoretical values of axial strains ($10^{-6}$) due to bending are $\varepsilon_{46b} = 7.0$ (1.1), $\varepsilon_{56b} = 1.5$ (0.3), and $\varepsilon_{57b} = 0.0$ (1.1).

The same strain gages that were used for static measurements were also used for dynamic measurements. A detailed analysis of the dynamic range of strain gages was given by Oi (1966), where it is shown that the rise-time $t_r$ of a gage to an ideal step function is given by

$$t_r < 0.2 \text{ ms} + 0.8L/c \tag{25}$$

where $L$ = length of the gage; and $c$ = wave speed of the substrate on which it is mounted. Furthermore, Oi shows that the "cutoff" frequency ($f_c$) of the gage, where its response falls to $1/\sqrt{2}$ of its static response, is given by

$$f_c = 0.35/t_r \tag{26}$$

Given that $L$ = 5.59 mm for the gage and $c$ = 5,039 m/s for $T$—6061 aluminum, one expects $f_c$ to be 318 kHz.

Fig. 3 shows the experimental setup to capture for recording and processing the data. The gages were connected to an eight-channel bridge amplifier circuit from Ectron Corporation, which has a 0.7 rolloff point of 100 kHz. The signals from the amplifier were captured digitally using a Tektronix 390AD converter that was set to sample at 1 MHz, that is, 10 times the bandwidth of the amplifier. A total of eight channels could be recorded simultaneously with 2,048 data points in each channel. Typically, signals from two gages mounted on either side of four members were recorded simultaneously. Dynamic axial strains were determined by averaging the signals from both sides of a member. The experimental error is estimated as 1 μ strain, and appears to be due to random thermal noise.

The truss was loaded first statically by hanging a known weight with a string at the central joint. Dynamic loading was created by suddenly burning the string. The source time function of the transient excitation is therefore simulated by a step unloading or the complementary Heaviside step function, $1 - H(t)$. There are two advantages of using a step pulse. First, as mentioned previously, the static strain before unloading provides a way to check the static response of the structure. Second, in principle, such a sharp pulse makes it easier to discern individual wave arrivals. Several types of threads and polymer-based lines were tried, but a braided trolling line (rated for 50-lb loads) was found to give the sharpest pulse, producing a steplike pulse of 20 to 30 μs duration when it was loaded with 113 N and burned with a cigarette lighter.

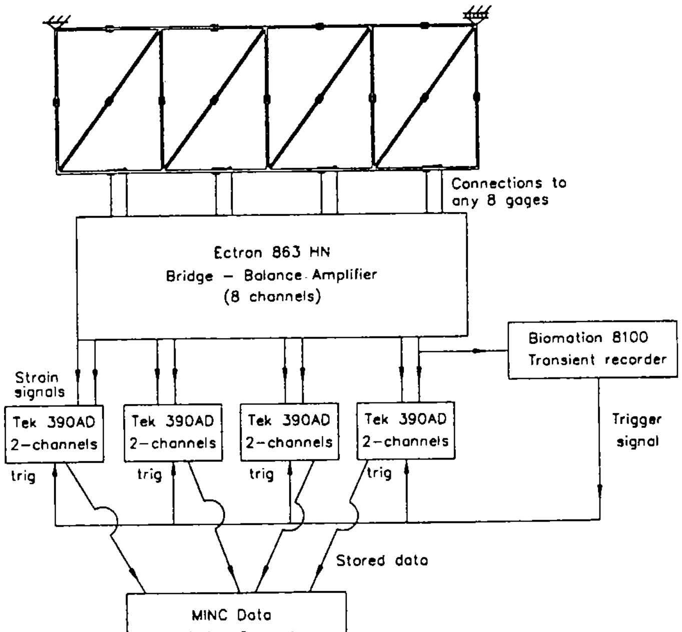


**FIG. 3. Schematic of Apparatus Used for Data Acquisition**

Experimental strain data due to axial force for the aforementioned three members as shown are discussed along with the theoretical transient results in the next section. The step unloading dynamic data are converted to those for the step loading, a unit Heaviside time function $H(t)$. The conversion was achieved by subtracting the experimental data for the complemental Heaviside function from a constant value of the static initial strain.

## TRANSIENT AXIAL WAVES AND EFFECT OF BENDING BY JOINTS

The method of reverberation matrix presented in previous sections has been implemented in FORTRAN code, with the frequency synthesis implied by (24) accomplished via the Fast Fourier Transform (FFT). As a check, results were compared against those obtained by ray-tracing of D'Alembert's solution (Boley and Chao 1957) and found to be in excellent agreement, as shown in Fig. 4 for member 1-2 as a typical example. The theoretical results are shown for a unit step compressive force (Heaviside step function) applied to joint 6.

In Fig. 4 and subsequent figures, the strain in each member is normalized by $\varepsilon_o = f_6/EA$ = 40.6 μm/m, where $f_6$ = weight at joint 6, 113 N; and $EA = 2.782 \times 10^6$ N for the square aluminum rod. Because the dimensions of each bay are in a 3:4:5 ratio, the time scale is normalized so that the time for an axial wave to traverse each corresponding member is 3, 4, and 5 time units, respectively—this makes one normalized time unit 20.16 μs.

Dynamic axial strains in the members around the loaded

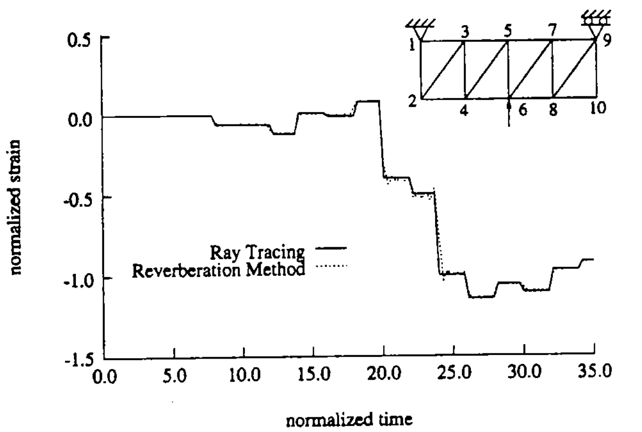


**FIG. 4. Comparison of Dynamic Strains as Determined by Ray Tracing and Reverberation Method in Member 1-2**

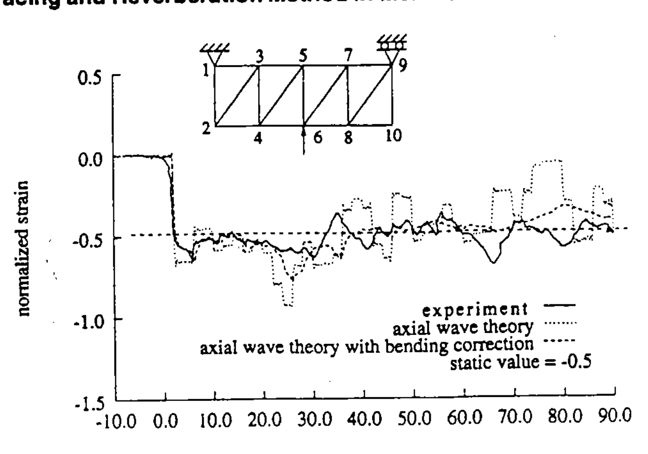


**FIG. 5. Axial Strain in Member 6-5**

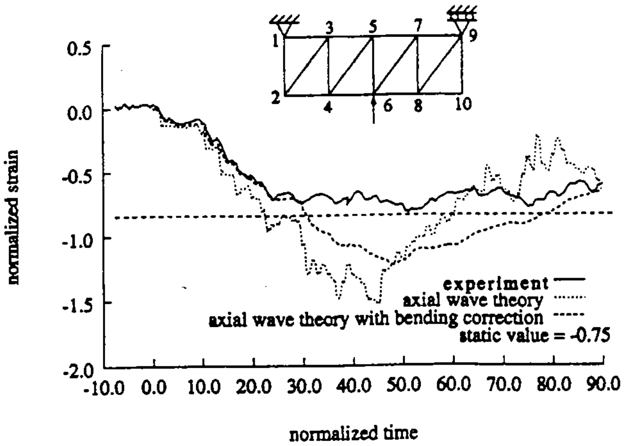


**FIG. 6. Axial Strain in Member 4-6**

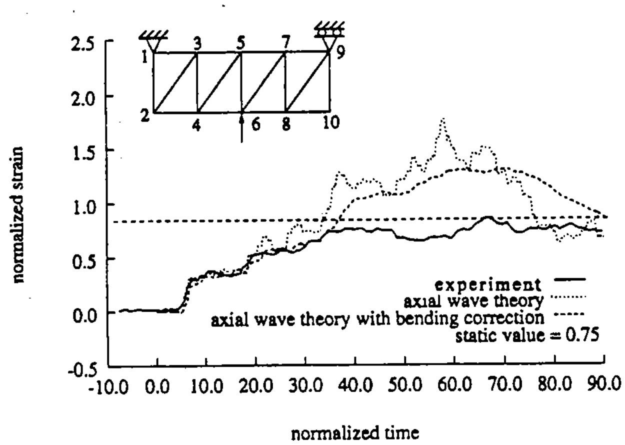


**FIG. 7. Axial Strain in Member 5-7**

joint for both experimental data and theoretical values, using the theory of the previous sections and the modified theory to be described below, are shown in Figs. 5–7. It is notable that the measured first arrivals, instead of being sharp step functions, are more ramplike, with a rise time usually much longer than the 20 to 30 μs of the source functions. This is in fact due to dispersion in the axial scattering coefficients brought about by the influence of bending forces. To include the effect of these bending forces on the axial waves it is necessary to modify the previously described theory.

## Modified Scattering Matrix

The scattering of incident axial wave or flexural wave at a rigid joint has been discussed by several authors (Howard 1990; Desmond 1981; Doyle and Kamle 1987). Scattering coefficients may be derived under the assumption that members support two transverse (or bending) modes corresponding to the Timoshenko theory or the Euler-Bernoulli theory of beams. Altogether, three wave modes propagate along the members independently of each other: one axial mode and the two transverse modes. These three modes are coupled only at the joints by equations that express (1) balance of forces and moments; (2) compatibility of motion of the members; and (3) constraint conditions expressing the manner in which the members are joined together, such as whether the joint is a pinned joint or rigid joint. The mathematical derivation is referred to Howard (1990); the scattering coefficient for either pinned-joint or rigid joint is stated in the following form:

$$S^{MJK} = \frac{A_2^{MJK}\sigma^2\omega + A_1^{MJK}\sigma\sqrt{\omega} + A_0^{MJK}}{B_2^J\sigma^2\omega + B_1^J\sigma\sqrt{\omega} + B_0^J} \tag{27}$$

$$s^{JM} = \frac{C_2^{JM}\sigma^2\omega + C_1^{JM}\sigma\sqrt{\omega} + C_0^{JM}}{B_2^J\sigma^2\omega + B_1^J\sigma\sqrt{\omega} + B_0^J} \tag{28}$$

where $\sigma^{JQ} = [(R/c)^{JQ}]^{1/2}$; and $R$ = radius of gyration of the cross section of member $JQ$. As $\sigma\sqrt{\omega}$ goes to zero, they approach the following limits, respectively:

$$\lim_{\sigma\sqrt{\omega}\to 0} S^{MJK} = \frac{A_0^{MJK}}{B_0^J} \tag{29}$$

$$\lim_{\sigma\sqrt{\omega}\to 0} s^{JM} = \frac{C_0^{MJK}}{B_0^J} \tag{30}$$

This limit is precisely the solution obtained for axial waves alone. This result can be further understood by observing that $\sigma\sqrt{\omega} = [2\pi(R/\lambda)^{JQ}]^{1/2}$, where $\lambda$ is the wavelength for axial waves in a member. Thus, when the radius of gyration of the truss members is much smaller than the wavelength of axial waves, the scattering may be described in terms of axial waves alone.

The model employed in the section on "Reverberation of Waves in Truss," which considers only axial stress waves, is a limiting case of a more general model which accounts for the coupling of axial and bending forces at a joint. As the frequency increases, this coupling increases and axial wave energy is increasingly converted into bending wave energy upon scattering at joints. Eqs. (27) and (28) thus represent an extension of the axial wave scattering coefficients that takes into account the conversion of axial wave energy to bending energy upon scattering. It is relatively straightforward to substitute (27) and (28) for the scattering coefficients used in (5), and thus account for this conversion in the reverberation method. However, a complete model would also describe mode conversion from bending to axial waves at joints, and the results based on the complete model will be reported in the future.

## Dynamic Responses in Chords and Vertical Member

Figs. 5, 6, and 7 show the experimental results as well as two sets of theoretical records for the dynamic strains in vertical member 5-6, and two chords 4-6 and 5-7, respectively. One set of the theoretical results is based on the axial wave theory in the section on "Reverberation of Waves in Truss," and the other on the modified theory with bending correction as given in this section. The original experimental data for the case of unit step unloading have been converted to the records for the case of unit step loading as explained in the previous section. As can be seen, the agreement of the modified theory with experiments is superior to that of the axial wave theory. In fact, up to about 30 time units the agreement is very good.

In all three figures the experimental records gradually approach the statical values after about 30 time units. The theoretical result for the vertical member (Fig. 5) shows a 50% overshot, and then oscillation about the static value. The experiment record shows a somewhat smaller dynamic overshot. The theoretical curves for chord 4-6 (Fig. 6) and chord 5-7 (Fig. 7), however, show a 46% and 55% respectively overshot, whereas the experiment records show very small dynamic amplifications.

There are two likely causes for this divergence between theory and experiment. The first is the neglect of mode conversion. Although the modified theory has taken into account the effect of rotation of a joint on the scattering coefficient (reflection and transmission) for the axial waves, it has not included the propagation of flexural waves in each member. The neglect of flexural waves in all members could introduce errors. The second cause is the oversimplification of the mathematical model for the actual support. This is suggested by the observation that discrepancy starts after the normalized time $t = 20$, which is about equal to the arrival time of all waves being scattered once by either one of the two supports. The discrepancy becomes large after $t = 30$, which is about equal to the arrival time for all waves that have been scattered twice at the support. It could probably be reduced by modifying the mathematical equations modeling the support such as (38).

## CONCLUSION

The dynamic response of planar trusses has been studied from a standpoint of the multiple reverberations of traveling stress waves in such structures. The proposed method of the reverberation matrix, which is an alternative to conventional matrix method, has been shown to be well suited for calculating these transients when the number of ray paths makes ray tracing of the D'Alembert solution impractical.

Due to the difficulty of fabricating an ideal pinned joint in the laboratory, the writers welded slender square aluminum bars to form the model truss. Experimental methods using a sensitive strain gage and a rapid unloading technique were developed to record the transient strains up to 40.6 μm/m in all members generated by a force of 113 N with a steplike time function. Wide-bandwidth electronic equipment was used to measure and record transient signal with a rise time of 30 μs and a duration of 2 ms.

The very early transient behavior of the experimental structure is described accurately by the theory of axial waves in a truss with pinned joints (Boley and Chao 1957). The theoretical results are improved by the theory of axial waves in a truss with rigid joints. It is expected that the remaining discrepancies between experimental and theoretical results could be further reduced by applying a general theory including both axial waves and flexural waves in the truss, and by modifying the mathematical equations for the hinged and roller support. It is nevertheless hoped that this investigation will be useful to analyze the impact response of trusses, for system identification by ultrasonic NDT, and for dynamic control of structural vibration.

## APPENDIX I. SCATTERING COEFFICIENTS FOR AXIAL WAVES ALONE

Consider Fig. 1. The equations of force balance are

$$\sum_{Q} F^{JQ}(0, t)\cos \theta^{JQ} = f_X = f(t)\cos \psi \ (Q = 1^J, 2^J, \ldots, m^J) \quad (31)$$

$$\sum_{Q} F^{JQ}(0, t)\sin \theta^{JQ} = f_Y = f(t)\sin \psi \quad (32)$$

Eq. (31) expresses force balance in the $x$-direction and (32) expresses force balance in the $y$-direction. The summation runs through all $m^J$ members connected to joint $J$.

The compatibility requirement arises from the assumption that one may treat the joint as a massless point. Then, the vector displacements of all members should be the same at their point of connection. If the joint displacement in the $x$-direction is denoted as $U^J$ and in the $y$-direction as $V^J$, the compatibility requirement may be expressed as

$$u^{JQ}(0, t) = U^J(t)\cos \theta^{JQ} + V^J(t)\sin \theta^{JQ} \ (Q = 1^J, 2^J, \ldots, m^J) \quad (33)$$

One may solve (31)–(33) by substituting (3) for $u^{JK}$ and using

$$F^{JK} = (EA)^{JK} \frac{\partial u^{JK}(x^{HK}, t)}{\partial x^{JK}} \quad (34)$$

where $A^{JK}$ = cross-sectional area of member $JK$. Substituting (3) and (34) into (31) and (32) one can express all departing waves $d^{JM}$ in terms of arriving waves $a^{JM}$ and applied forces $f^J$ as follows:

$$d^{JM}(\omega) = \sum_{Q} S^{MJQ}a^{JQ}(\omega) + s^{JM}(\omega) \ (Q = 1^J, 2^J, \ldots, m^J) \quad (35)$$

where

$$S^{MJK} = -\delta^{MK} + 4\,\frac{\sum_{Q} \gamma^{JQ}\gamma^{JK} \sin(\theta^{JM} - \theta^{JQ})\sin(\theta^{JK} - \theta^{JQ})}{\sum_{P}\sum_{L} \gamma^{JP}\gamma^{JL} \sin(\theta^{JP} - \theta^{JL})}$$

$$(Q, P, L = 1^J, 2^J, \ldots, m^J) \quad (36)$$

$$s^{JM} = -2i\,\frac{\sum_{Q} \gamma^{JQ} \sin(\psi^J - \theta^{JQ})\sin(\theta^{JM} - \theta^{JQ})}{\omega\sum_{P}\sum_{L} \gamma^{JP}\gamma^{JL} \sin^2(\theta^{JP} - \theta^{JL})}\,f(\omega) \quad (37)$$

where $\delta^{MK}$ = Kronecker delta (equal to 1 if $M = K$; 0 otherwise) and

$$\gamma^{JM} = A^{JM}\sqrt{\rho^{JM}E^{JM}}$$

If the hinged support is fixed, (35) or (6) is then determined from the equation for a fixed joint,

$$u^{JQ} = 0 \quad (38)$$

The reactive forces at the hinged support can then be calculated from (31) and (32). On the other hand, if the joint is supported by a roller free to slide in the $x$-direction, and the vertical reactive force $f_Y$ in (32) and $U^J$ are treated as unknowns, the scattering matrix is then solved by eliminating $f_Y$ and $U^J$ from the system of equations 931)–(33).


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## APPENDIX III. NOTATION

*The following symbols are used in this paper:*

$\mathbf{a}$ = global arrival vector;
$\mathbf{a}^J$ = total arrival vector at joint $J$;
$\mathbf{a}^{JK}$ = complex amplitude of wave arriving at joint $J$ from member $K$;
$c$ = axial wave velocity;
$\mathbf{d}$ = global departure vector;
$\mathbf{d}^J$ = total departure vector at joint $J$;
$d^{JK}$ = complex amplitude of wave departing joint $J$ toward member $K$;
$E$ = Young's modulus;
$F^{JQ}$ = axial force in member $JQ$;
$f_c$ = cutoff frequency for response of strain gage;
$f^j$ = force at joint $j$;
$\mathbf{I}$ = identity matrix;
$I, J, \ldots, Q$ = joint identifiers;
$\mathbf{K}$ = diagonal matrix of member wave numbers;
$k$ = wave number;
$\mathbf{L}$ = vector of member lengths;
$L$ = length of strain gage;
$M^{JQ}$ = bending moment in member $JQ$;
$m$ = number of members in entire structure;
$m^J$ = number of members attached to joint $J$;
$n$ = number of joints in entire structure;
$n^J$ = number of neighboring joints for joint $J$;
$\mathbf{P}$ = propagation matrix;
$\mathbf{R}$ = reverberation matrix;
$R$ = radius of gyration of member cross section;
$\mathbf{S}$ = global scattering matrix;
$\mathbf{S}^J$ = scattering matrix for joint $J$;
$S^{MJK}$ = scattering coefficient for member $MJ$ to member $JK$;
$\mathbf{s}$ = global source matrix;
$\mathbf{s}^J$ = source vector for joint $J$;
$s^{JM}$ = source wave emanating from joint $J$ in member $JM$;
$t$ = time;
$t_r$ = rise time response of strain gage;
$\mathbf{U}$ = permutation matrix for permuting departure vectors to arrival vectors;
$\mathbf{u}^{JK}$ = axial displacement in member $JK$;
$V^{JQ}$ = transverse force in member $JQ$;
$v^{JQ}$ = transverse displacement in member $JQ$;
$\mathbf{x}$ = vector of positions;
$x^{JK}$ = position along member $JK$;
$\theta^{JQ}$ = angle of member $JQ$ relative to reference;
$\rho$ = density;
$\sigma$ = square root of ratio of radius a member's cross-sectional radius to its axial wave speed;
$\psi$ = angle of applied force relative to horizontal reference frame;
$\omega$ = radial frequency; and
$\tilde{}$ = symbol for permutated vector.